

A SET OF CONDITIONS WHICH EVERY LOCALLY
CONNECTED DESIGN MUST SATISFY

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Abstract

The concept of globally and locally connected designs are introduced here. Our definition of locally connected block designs is the same as the connected designs of Bose. Connectedness is a desirable and important property which every reasonable block design should enjoy. This is so because every elementary treatment contrast is estimable if and only if the design is at least locally connected. In particular, if the design is globally connected then one can construct better (in the sense of minimum variance) estimators for all the elementary contrasts. This paper provides some simple algorithms for excluding nonlocally connected designs. Several rules of Chakrabarti for establishing locally connectedness of the given design are also given. Some unsolved problems are stated.

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0. Summary. The concept of globally and locally connected designs are introduced here. Our definition of locally connected block designs is the same as the connected designs of Bose. Connectedness is a desirable and important property which every reasonable block design should enjoy. This is so because every elementary treatment contrast is estimable if and only if the design is at least locally connected. In particular, if the design is globally connected then one can construct better (in the sense of minimum variance) estimators for all the elementary contrasts. This paper provides some simple algorithms for excluding nonlocally connected designs. Several rules of Chakrabarti for establishing locally connectedness of the given design are also given. Some unsolved problems are stated.

1. Introduction. Let $\Omega = \{a_1, a_2, \dots, a_v\}$ be a set of v treatments. By a block design with parameters $v, b; r_1, \dots, r_v; k_1, k_2, \dots, k_b$, and incidence structure N denoted by $BD(v, b; r_1, \dots, r_v; k_1, \dots, k_b, N)$ on Ω we shall mean an allocation of elements of Ω one on each of the $m = k_1 + \dots + k_b$ experimental units arranged in b blocks or groups of experimental units designated by $B_j, j=1, 2, \dots, b$ with B_j of size k_j such that a_i is assigned into r_i experimental units. Thus $r_1 + \dots + r_v = k_1 + \dots + k_b$. Note that B_j is not necessarily a subset of Ω . $N = (n_{ij})$ is the usual $v \times b$ incidence matrix of the design where n_{ij} denotes the number of experimental units in the j^{th} block receiving the i^{th} treatment. In the sequel D denotes a general block design.

2. Background materials and the results.

Definition. Two treatments a_i and a_j , $i \neq j$ in D are said to be globally connected if for any replication of a_i and a_j it is possible to construct a chain of treatments beginning with a_i and ending with a_j such that every consecutive pair of treatments in the chain occurs together in a block. Two treatments a_i and a_j are said to be locally connected if we can find a replication of a_i and a_j in D provided that it is possible to construct a chain of treatments beginning with a_i and ending with a_j such that every consecutive pair of treatments in the chain occurs together in a block. The relationship a_i globally (locally) connected to a_j defines an equivalence relation on D which induces disjoint equivalence classes. D is said to be globally (locally) connected if there is only one equivalence class, i.e., if every pair of treatments is globally (locally) connected. It should be mentioned that Bose's [1] definition of connected design is the same as our locally connected design.

The following well-known theorem indicates the desirability and the importance of connected designs.

Theorem 2.1. All elementary treatment contrasts in D are estimable if and only if D is at least locally connected.

It can be shown that if in particular D is globally connected then every replication of a_i and a_j , $i \neq j$, $i, j = 1, 2, \dots, v$ can participate in estimating the contrast between the effect of a_i and a_j . Therefore, if D_1 and D_2 are two designs with the same parameters $v, b; r_1, \dots, r_v; k_1, k_2, \dots, k_b$ on Ω then one can construct better (in the sense of minimum variance) estimators of all elementary contrasts in D_1 than in D_2 if D_1 is globally connected while D_2 is only locally connected.

In this note we limit ourselves to the locally connected designs. The corresponding problems associated with globally connected designs will be stated as unsolved problems.

Theorem 2.1 has led many authors, as in the case of variance balanced designs, to characterize locally connected designs through its coefficient matrix, viz.

$$C = [c_{ij}] = R - NK^{-1}N'$$

where $R = \text{diag}[r_1, \dots, r_v]$, $K^{-1} = \text{diag}[k_1^{-1}, \dots, k_b^{-1}]$ and N' is the transpose of N . Since every row and column in C adds up to zero thus the rank of C is at most $v-1$. Indeed, it can be shown that C is a positive semidefinite matrix. (For the importance of the matrix C in the theory of block designs see Chakrabarti [2,3], John [4], and Kempthorne [5]) The following known theorem characterizes locally connected designs through their C matrices.

Theorem 2.2. D is locally connected if and only if its C matrix is of rank $v-1$.

Even though this is a mathematically neat result, the problem is that in general it is not easy to compute the rank of C for v large. Therefore, what we really need is a simple and straightforward algorithm. With this in mind we prove the following.

Theorem 2.3.

(a) D is not locally connected if there is any zero or negative element on the main diagonal of its C matrix.

(b) D is not locally connected if there is an i and j , $i \neq j$ such that in its C matrix we have $c_{ii}c_{jj} \leq c_{ij}^2$ and $v > 2$.

(c) D is not locally connected if the largest element (in absolute sense) in its C matrix does not lie on its main diagonal and $v > 2$.

Lemma 2.1. Let A be an $n \times n$ symmetric positive semidefinite matrix of rank $n-1$. Then $X'AX = 0$ if and only if X is in the subspace generated by the eigen vector associated with the zero eigen value of A.

Proof of Theorem 2.3:

(a) Suppose D is locally connected and there is an i such that $c_{ii} < 0$. We prove that this is impossible. Since C is a positive semidefinite then

$$X'CX \geq 0 \quad (1)$$

for every nonzero vector X. In particular let X_1 be the vector with 1 in its i^{th} component and zero elsewhere. Then

$$X_1'CX_1 = c_{ii} \geq 0$$

by (1). Therefore c_{ii} can be at least zero. Now we show that $c_{ii} = 0$ is an impossible case. For otherwise,

$$X_1'CX_1 = 0$$

implies that X_1 is in the subspace generated by $\xi = (1, 1, \dots, 1)$, the eigen vector associated with zero eigen value of C. But this is impossible since X_1 and ξ are independent. Chackrabarti [2] has also obtained this result in a different way.

(b) Suppose D is locally connected and there exists an i and a j , $i \neq j$ such that $c_{ii}c_{jj} \leq c_{ij}^2$. We shall prove that this is also impossible. Let X_2 be any vector whose components satisfy the following conditions

- (i) $x_{2j} > 0$,
- (ii) $x_{2i} = -c_{ij}x_{2j}/c_{ii}$,
- (iii) all other components arbitrary.

Since X_2 is independent of ξ then

$$X_2' C X_2 > 0. \quad (2)$$

But

$$\begin{aligned} X_2' C X_2 &= c_{ii}x_{2i}^2 + 2c_{ij}x_{2i}x_{2j} + c_{jj}x_{2j}^2 \\ &= x_{2j}(c_{ii}c_{jj} - c_{ij}^2)/c_{ii} < 0 \end{aligned}$$

by assumption and (a). But this contradicts (2).

(c) Suppose D is locally connected and for some i and j we have $|c_{ij}| \geq c_{ii}$ and $|c_{ij}| \geq c_{jj}$ where from (a) we know that $c_{ii} > 0$, $c_{jj} > 0$. In this case $c_{ij}^2 \geq c_{ii}c_{jj}$ which contradicts (b).

The following theorem proved by Chakrabarti [3] helps in many cases to establish locally connectedness of D .

Theorem 2.4.

- (a) D is locally connected if every element in its C matrix is different from zero.
- (b) D is locally connected if its C matrix contains a row (column) of non-zero elements.
- (c) D is locally connected if there is at least one non-zero element in row i , $i=1,2,\dots,v-1$ above the non-zero elements in the last row of its C matrix.
- (d) D is locally connected if there are more than $v-t$ non-zero elements in row i , $i=1,2,\dots,v-1$ and there are only t non-zero elements in the last row of its C matrix.

3. Conclusion. Of course Theorems 2.3 and 2.4 do not exhaust all the necessary and sufficient conditions. It will be interesting if one could add more simple conditions to these two theorems. Also, since the matrix N or NN' completely resemble the combinatorial structure of D , one should be able to find a set of simple necessary and sufficient conditions on N or NN' which guarantees the locally connectedness of D . The solution of the same problems even in terms of the C matrix are not known for globally connected designs.

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10/14/71